1.1 Systems of Linear Equations

Practice: 3, 7, 12, 16, 18, 23, 24, 29

Required: 25, 28, 33

1.1.1 Linear equations

A linear equation:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

EXAMPLE 1 :

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

$\downarrow$ rearranged $\downarrow$ $\qquad$ $\downarrow$ rearranged $\downarrow$

$$3x_1 - 5x_2 = -2 \qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

Not linear:

$$4x_1 - 6x_2 = x_1x_2 \qquad \text{and} \qquad \underline{\hspace{3cm}}.$$

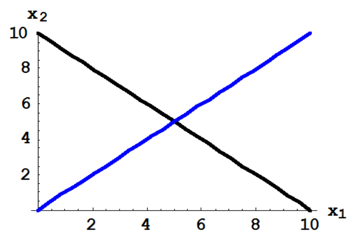
1.1.2 A system of linear equations (linear system):

A system of linear equations is a collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, ..., x_n$.

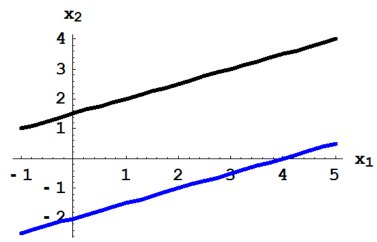
A **solution** of a linear system is a list $(s_1, s_2, ..., s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, ..., s_n$ are substituted for $x_1, x_2, ..., x_n$, respectively.

EXAMPLE 2 : Two equations in two variables:

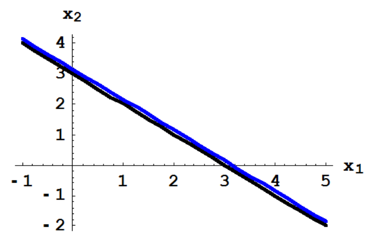
$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 10 \\ -x_1 & + & x_2 & = & 0 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array}$$



one unique solution



no solution



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Theorem 1 *A system of linear equations has either*

1. *exactly one solution (consistent) or*
2. *infinitely many solutions (consistent) or*
3. *no solution (inconsistent).*

EXAMPLE 3 : Three equations in three variables. Each equation determines a plane in 3-space. In this case we have several options:

1. The planes intersect in one point.
2. Infinitely many solutions: the planes intersect in ______________________.
3. Infinitely many solutions: the planes intersect in ______________________.
4. The planes do not intersect.

The **solution set:**

- The set of all possible solutions of a linear system.

Equivalent systems:

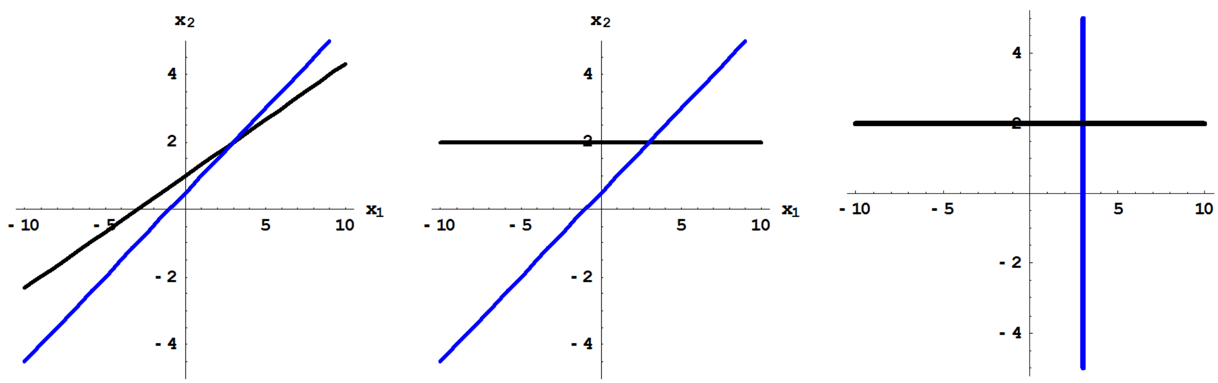
- Two linear systems with the same solution set.

1.1.3 Strategy for solving a linear system

- *Replace one system with an equivalent system that is easier to solve.*

EXAMPLE 4 :

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \longrightarrow \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \longrightarrow \begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array}$$



$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \qquad \begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array}$$

1.1.4 Matrix Notation

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \underset{\text{(coefficient matrix)}}{\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}} \qquad \underset{\text{(augmented matrix)}}{\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix}}$$

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \qquad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

1.1.5 Elementary Row Operations:

1. *(Replacement)* Add one row to a multiple of another row ($R_i = R_i + \alpha R_j$).
2. *(Interchange)* Interchange two rows ($R_i \leftrightarrow R_j$).
3. *(Scaling)* Multiply all entries in a row by a nonzero constant ($R_i = \beta R_i, \quad \beta \neq 0$).

Row equivalent matrices: Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

EXAMPLE 5 :

i. $$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \qquad R_3 = R_3 + 4R_1$$

ii $$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \qquad R_2 = \_\_\_\_\_\_\_\_$$

iii $$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix} \qquad R_3 = \_\_\_\_\_\_\_\_$$

iii $$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \begin{array}{l} R_1 = \_\_\_\_\_\_\_\_ \\ R_2 = \_\_\_\_\_\_\_\_ \end{array}$$

iv $$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad R_1 = \_\_\_\_\_\_\_\_$$

iv $$\begin{array}{rrrcr} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solution: $(29, 16, 3)$

Check: Is $(29, 16, 3)$ a solution of the ***original*** system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcrcr} (29) - 2(16) + \quad 3 & = & 29 - 32 + 3 & = & 0 \\ 2(16) - 8(3) & = & 32 - 24 & = & 8 \\ -4(29) + 5(16) + 9(3) & = & -116 + 80 + 27 & = & -9 \end{array}$$

1.1.6 Two Fundamental Questions (Existence and Uniqueness)

1) Is the system consistent; (i.e. does a solution **exist**?)

2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

EXAMPLE 6 : Is this system consistent?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

In the last example, this system was reduced to the triangular form:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This is sufficient to see that the system is consistent and unique. Why?

EXAMPLE 7 : Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix}$$

Solution: Row operations produce:

$$\begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \quad \leftarrow \textit{Never true} \end{aligned}$$

The original system is ___________!

EXAMPLE 8 : For what values of h will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

Solution: Reduce to triangular form.

$$\begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h + \frac{8}{3} \end{bmatrix}$$

The second equation is ___________. The system is consistent only if ________________.