## 5.1 Eigenvectors & Eigenvalues

4,9, 13,19,22,25, 33

The basic concepts presented here - *eigenvectors* and *eigenvalues* - are useful throughout pure and applied mathematics. Eigenvalues are also used to study difference equations and *continuous* dynamical systems. They provide critical information in engineering design, and they arise naturally in such fields as physics and chemistry.

**EXAMPLE 1 :** Let $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Examine the images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by $A$.

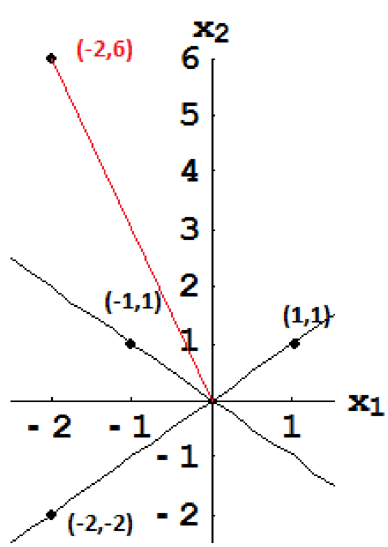
*Solution*

$$A\mathbf{u} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \neq \lambda\mathbf{v} \text{ for any scalar } \lambda.$$

$\mathbf{u}$ is called an *eigenvector* of $A$.

$\mathbf{v}$ is not an eigenvector of $A$ since $A\mathbf{v}$ is not a multiple of $\mathbf{v}$.




$A\mathbf{u} = -2\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$

**Definition 1** *An **eigenvector** of an $n \times n$ matrix $A$ is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda$. A scalar $\lambda$ is called an **eigenvalue** of $A$ if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to $\lambda$.*

**EXAMPLE 2 :** Show that 4 is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

*Solution:* Scalar 4 is an eigenvalue of $A$ if and only if $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution.

$$A\mathbf{x} - 4\mathbf{x} = \mathbf{0}$$
$$A\mathbf{x} - 4\,(\_\_\_\_)\,\mathbf{x} = \mathbf{0}$$
$$(A - 4I)\,\mathbf{x} = \mathbf{0}.$$

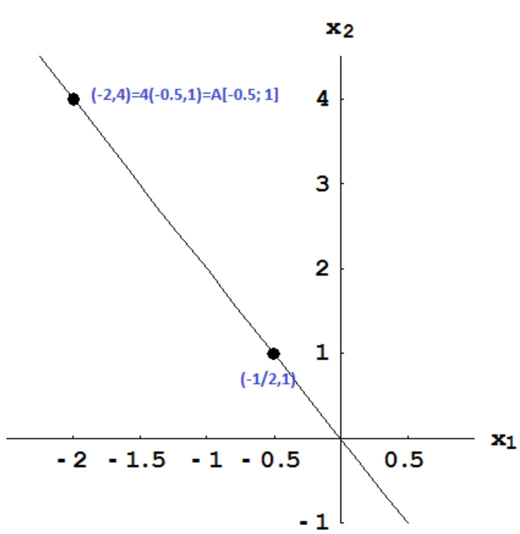
To solve $(A - 4I)\,\mathbf{x} = \mathbf{0}$, we need to find $A - 4I$ first:

$$A - 4I = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}$$

Now solve $(A - 4I)\,\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -4 & -2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{x} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} = x_2 \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

Each vector of the form $x_2 \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 4$.




Eigenspace for $\lambda = 4$

**Definition 2** *The set of all solutions to* $(A-\lambda I)\,\mathbf{x} = \mathbf{0}$ *is called the* ***eigenspace*** *of* $A$ *corresponding to* $\lambda$.

**EXAMPLE 3 :** Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$. An eigenvalue of $A$ is $\lambda = 2$. Find a basis for the corresponding eigenspace.

*Solution:*

$$A-2I = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} \_\_\_ & 0 & 0 \\ 0 & \_\_\_ & 0 \\ 0 & 0 & \_\_\_ \end{bmatrix}$$

$$= \begin{bmatrix} 2-\_\_\_ & 0 & 0 \\ -1 & 3-\_\_\_ & 1 \\ -1 & 1 & 3-\_\_\_ \end{bmatrix} = \begin{bmatrix} \_\_\_ & 0 & 0 \\ -1 & \_\_\_ & 1 \\ -1 & 1 & \_\_\_ \end{bmatrix}$$

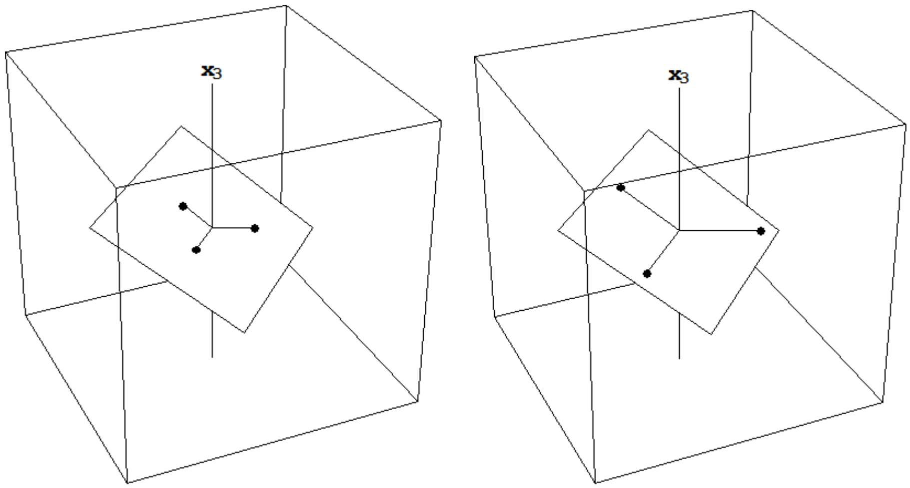
Augmented matrix for $(A-2I)\,\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ x_2 \\ x_3 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \_\_\_\_ \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

So a basis for the eigenspace corresponding to $\lambda = 2$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$




Effects of Multiplying Vectors in Eigenspaces for $\lambda = 2$ by $A$

**EXAMPLE 4 :** Suppose $\lambda$ is eigenvalue of $A$. Determine an eigenvalue of $A^2$ and $A^3$. In general, what is an eigenvalue of $A^n$?

*Solution:* Since $\lambda$ is eigenvalue of $A$, there is a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then

$$\_\_\_A\mathbf{x} = \_\_\_\lambda\mathbf{x}$$

$$A^2\mathbf{x} = \lambda A\mathbf{x}$$
$$A^2\mathbf{x} = \lambda\_\_\_\mathbf{x}$$
$$A^2\mathbf{x} = \lambda^2\mathbf{x}$$

Therefore $\lambda^2$ is an eigenvalue of $A^2$.

Show that $\lambda^3$ is an eigenvalue of $A^3$:

$$\_\_\_A^2\mathbf{x} = \_\_\_\lambda^2\mathbf{x}$$
$$A^3\mathbf{x} = \lambda^2 A\mathbf{x}$$
$$A^3\mathbf{x} = \lambda^3\mathbf{x}$$

Therefore $\lambda^3$ is an eigenvalue of $A^3$.

In general, ______ is an eigenvalue of $A^n$.

**Theorem 1** *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

**Proof**

*Proof for the 3×3 Upper Triangular Case*: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

and then

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}.$$

By definition, $\lambda$ is an eigenvalue of $A$ if and only if $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ has a nontrivial solution. This occurs if and only if $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ has a free variable.
When does this occur?

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**Theorem 2** *If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix $A$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a linearly independent set.*

**EXAMPLE 5 :** Let $A == \begin{bmatrix} -1 & 2 & 3 \\ 0 & -4 & 5 \\ 0 & 0 & -6 \end{bmatrix}$. How many linearly independent eigenvectors does $A$ have?
*Solution:* The eigenvalues of $A$ are ______________________.

Therefore, $A$ has ___________linearly independent eigenvectors.