5.2 The Characteristic Equation

5.2: 3, 9, 18, 21

Review: In order to find eigenvalues we need

$$A \quad \mathbf{x} \quad = \quad \lambda \quad \mathbf{x}$$

Find eigenvectors $\mathbf{x}$ by solving $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$.

How do we find the eigenvalues λ**?**

$$\mathbf{x} \text{ must be nonzero}$$

$$\Downarrow$$

$$(A - \lambda I)\,\mathbf{x} = \mathbf{0} \text{ must have nontrivial solutions}$$

$$\Downarrow$$

$$(A - \lambda I) \text{ is not invertible}$$

$$\Downarrow$$

$$\det(A - \lambda I) = 0$$

(called the *characteristic equation*)

$$\boxed{\text{Solve } \det(A - \lambda I) = 0 \text{ for } \lambda \text{ to find the eigenvalues.}}$$

Characteristic polynomial: $\det(A - \lambda I)$

Characteristic equation: $\det(A - \lambda I) = 0$

EXAMPLE 1 : Find the eigenvalues and corresponding basis for the eigenspaces of $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

Solution: Since

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -6 & 5 - \lambda \end{bmatrix},$$

the equation $\det(A - \lambda I) = 0$ becomes

$$-\lambda(5-\lambda)+6=0$$
$$\lambda^2-5\lambda+6=0$$

Factor:

$$(\lambda-2)(\lambda-3)=0.$$

So the eigenvalues are 2 and 3.

$(\lambda=2)$: We need to solve $(A-2I)\mathbf{x}=\mathbf{0}$, so we obtain the augmented matrix

$$\begin{bmatrix} \_\_\_ & 1 & 0 \\ -6 & \_\_\_ & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and a basis for the eigenspace is therefor the vector $\mathbf{v}_{\lambda=2}=\begin{bmatrix} \quad \\ \quad \end{bmatrix}$.

$(\lambda=3)$: We need to solve $(A-3I)\mathbf{x}=\mathbf{0}$, so we obtain the augmented matrix

$$\begin{bmatrix} \_\_\_ & 1 & 0 \\ -6 & \_\_\_ & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and a basis for the eigenspace is therefor the vector $\mathbf{v}_{\lambda=3}\begin{bmatrix} \quad \\ \quad \end{bmatrix}$.

Note that $\mathbf{v}_{\lambda=2}$ and $\mathbf{v}_{\lambda=3}$ are linearly independent. □

For a 3×3 matrix or larger, recall that a determinant can be computed by cofactor expansion.

EXAMPLE 2 : Find the eigenvalues of $A=\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

Solution:

$$A-\lambda I=\begin{bmatrix} 1-\_\_\_\_ & 2 & 1 \\ 0 & -5-\_\_\_\_ & 0 \\ 1 & 8 & 1-\_\_\_\_ \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & -5-\lambda & 0 \\ 1 & 8 & 1-\lambda \end{vmatrix} = (-5-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (-5-\lambda)\left[(1-\lambda)^2 - 1\right] = (-5-\lambda)\left[1 - 2\lambda + \lambda^2 - 1\right]$$

$$= \underline{\hspace{6cm}} = 0$$

$$\Rightarrow \lambda = -5, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}.$$

Theorem 1 *(The Invertible Matrix Theorem - continued)*

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. *The number* 0 *is not an eigenvalue of* A*.*

t. $\det A \neq 0$

Definition 1 *The (**algebraic**) **multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation.*

EXAMPLE 3 : Find the characteristic polynomial of

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 9 & 1 & 3 & 0 \\ 1 & 2 & 5 & -1 \end{bmatrix}$$

and then find all the eigenvalues and the algebraic multiplicity of each eigenvalue.

Solution:

Since A is a triangular matrix, by simple inspection we have that the eigenvalues are: ___ (algebraic multiplicity ___), ___ (algebraic multiplicity ___) and ___ (algebraic multiplicity ___).

EXAMPLE 4 : Matrices are used to store information on population growth. Call the proportion of the population that survives from one season to the next the survivability and the number of offspring the fecundity. A Leslie matrix stores this information in a precise and well-defined way. So, for example considering a bird population, you can write down a Leslie matrix for the hatchlings and adults in this population along the lines

$$\begin{bmatrix} \text{average fecundity of hatchlings next year} & \text{average fecundity of adults next year} \\ \text{survivability of hatchlings} & \text{survivability of adults.} \end{bmatrix}$$

Suppose the Leslie matrix for a particular population is given by $M = \begin{bmatrix} .95 & .90 \\ .05 & .10 \end{bmatrix}$.
So, if currently we have a population of birds of h hatchlings and a adults, the next year we will have

$$0.95(h) + 0.9(a) \text{ new hatchlings,}$$

$$0.05(h) + 0.1(a) \text{ adults.}$$

If we call $\mathbf{x_0} = \begin{bmatrix} h \\ a \end{bmatrix}$ the current **population vector**, the above expression is equivalent to find the population vector for the next year, which we will call $\mathbf{x_1}$ and we have

$$\mathbf{x_1} = \begin{bmatrix} .95 & .90 \\ .05 & .10 \end{bmatrix} \begin{bmatrix} h \\ a \end{bmatrix} = M\mathbf{x_0}.$$

Let's show now that if $\mathbf{x_0} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, then the population

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$$

converges to a steady state vector $\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.
The answer lies in examining the corresponding eigenvectors.
First we find the eigenvalues:

$$\det(M - \lambda I) = \det\left(\begin{bmatrix} .95 - \lambda & .90 \\ .05 & .10 - \lambda \end{bmatrix}\right) = \lambda^2 - 1.05\lambda + 0.05$$

So solve

$$\lambda^2 - 1.05\lambda + 0.05 = 0$$

By factoring

$$\lambda = 0.05,\ \lambda = 1$$

It can be shown that the eigenspace corresponding to $\lambda = 1$ is span$\{\mathbf{v}_1\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the eigenspace corresponding to $\lambda = 0.05$ is span$\{\mathbf{v}_2\}$ where $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Then for a given vector $\mathbf{x}_0$,

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

$$\mathbf{x}_1 = M\mathbf{x}_0 = M\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2\right) = c_1M\mathbf{v}_1 + c_2M\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\left(0.05\right)\mathbf{v}_2$$

$$\mathbf{x}_2 = M\mathbf{x}_1 = M\left(c_1\mathbf{v}_1 + c_2\left(0.05\right)\mathbf{v}_2\right) = c_1M\mathbf{v}_1 + c_2\left(0.05\right)M\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\left(0.05\right)^2\mathbf{v}_2$$

and in general

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2\left(0.05\right)^k\mathbf{v}_2$$

$$\text{and so } \lim_{k\to\infty}\mathbf{x}_k = \lim_{k\to\infty}\left(c_1\mathbf{v}_1 + c_2\left(0.05\right)^k\mathbf{v}_2\right) = c_1\mathbf{v}_1$$

Now, in our case

$$\mathbf{x_0} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

therefore $c_1 =$ ___, $c_2 =$ ___ and

$$c_1\mathbf{v}_1 = \underline{\quad}\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

This means that after a long period of time, we can expect an stable population of 2 hatchlings and 2 adults per year.