Discrete Mathematics - Test 2 Review - Fall 2010

1. You are on an interview at Google, and you overhear some managers brainstorming about how best to implement algorithms over a quaternary alphabet (using symbols 0, 1, 2, 3). The quaternary alphabet can be used to protect binary data from certain errors, however the simultaneous use of 0s and 3s introduces inter-symbol interference. One easy fix is to prohibit what are called "off runs" of 1's. In layman's terms, this means that in any quaternary string of length n, there must be an even number of 1's. You realize that your input could help pull Google out of its financial slump, so you pop down to offer some assistance. (Note that this definition of "off runs" is phoney, but it makes for a good story.)

    (a) Find a recurrence relation for q_n, the number of quaternary strings of length n that have an even number of 1's; clearly explain and justify all of your steps and be sure to include the right number of initial conditions.

    (b) Using your recurrence relation and the principle of mathematical induction, prove that $q_n = 2^{n-1} + 2(4^{n-1})$ for all $n \geq 2$.

2. You continue walking around the Google campus noticing that it is indeed true that all restaurants on campus are free and employees also enjoy complimentary recreational facilities. You want to get yourself noticed, so you cruise over to the grassy knoll where several managers are discussing the delivery of Google webpages. Fast delivery has become increasingly important with the launch of Google Maps and competing site Bing, so Google has hired Akamai Technologies to deliver their web content using servers networked around the globe. The Googlite managers are doing a cost-benefit analysis to determine whether to stick with Akamai's service or whether to acquire the necessary hardware to perform the distributed hosting on their own. Their decision will require knowledge concerning connectivity of networks. The managers are having a hard time keeping their logic straight while analyzing the networks. They keep assuming that what they hope to be true is already true. This is no way to perform an analysis! A mistake like this could cost Google millions! You swoop in just in time to help with the following:

    (a) Consider a graphical representation of a network that contains n vertices such that each vertex is adjacent to exactly k other vertices, where k is an odd, positive integer. This graphical representation has no loops or multiple edges; it is quite simple. Is it true that the total number of edges in this network is a multiple of k? If not, give an example to illustrate. If yes, prove it from scratch. Your airtight proof should have every step fully explained and fully justified.

    (b) Consider a different graphical model for a network with n vertices such that for any two distinct vertices x and y, the degree of x plus the degree of y is greater than or equal to $n - 1$ (symbolically, $\deg(x) + \deg(y) \geq n - 1$).

Again, there are no loops or multiple edges in their simple model. Must G be connected? If not, give an example to illustrate. If yes, give an airtight proof, where each step of the proof is fully explained and fully justified.

3. Is it possible to have a simple graph with $n \geq 3$ vertices, where each vertex has degree greater than or equal to $(n-1)/2$, that does not have a Hamilton circuit? If yes, find an example. If no, explain why not.

4. You are heading to Norway to check out the famous fjords. One wrong left turn later, and you find yourself overlooking a large group of short men debating the "fun factor" of a strange new Lego game. The Lego game can be played with a varying number of Legos, and the short men are trying to decide how many Legos should be included in the new game, to maximize the 'fun factor." The game begins with an initial configuration of Legos which can be best modelled by a graph. In the graph, the edges are Legos and the vertices are points where Legos connect together. Let $G(n)$ be a graphical model of a version of the game where $G(n)$ includes $2n$ vertices labelled $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ such that the vertices in the following order make a cycle $x_1, x_2, \ldots, x_n, y_n, y_{n-1}, \ldots, y_1$, (where y_1 is adjacent to x_1 to complete the cycle) and also x_i is adjacent to y_i for all i, $2 \leq i \leq n-1$. (Note that due to the described cycle, it is actually true that x_i is adjacent to y_i for all i, $1 \leq i \leq n$. The graph resembles a ladder, or a row of squares.) In this configuration, every vertex has degree 3 except for x_1, x_n, y_1, and y_n, which each have degree 2. Once this ladder-like configuration of Legos is in place, the game begins as follows. The player (yes, only one player) must select n of the Legos from the Legos in the initial configuration $G(n)$, with the added restriction that no two chosen Legos (edges) share a common connection (vertex). If the player can do this, s/he wins. For $n \geq 2$, let a_n denote the number of ways one can select n of the edges in $G(n)$ so that no two chosen edges share a common vertex. Find a recurrence relation for a_n. If the short men want to ensure that there are at least 20 ways to play the game, which model $G(n)$ should they use? Be sure to clearly explain and justify all of your work.

5. Your friend Hermione has arrived for her annual November vacation in Needham, Massachusetts. You are glad to see her, but you are beginning to wonder if it was such a good idea to have her come before Thanksgiving break. After all, you are owned by Olin this week and Hermione keeps taking up your brain space by stumping you with her shenanigans. For example, the night she arrived, she poured out ten shiny black stones from a red velvet pouch. She asked you to play a simple game - each player alternates turns, and during each turn, a player can remove either one or two stones. The winner is the player who removes the final stone(s). Since she is the guest, you let her go first. She keeps winning every time. You start to get suspicious of the red velvet bag. Did she sprinkle winning dust in her velvet bag? She goes to the bathroom and you inspect her bag for signs of winning dust. You do not notice any dust and spend the rest of the evening wondering how Hermione can be this lucky. After

many hours of pondering, you remember that graph theory has been useful for solving important problems involving practical situations such as the foraging patterns of mice, conundrums faced by the the soccer ball production industry, and the transport of wolves, goats, and cabbage.

(a) Surely you can use a graph to analyze this game to find out for certain if Hermione is using winning dust or if she has figured out a strategy guaranteed to win every time. Present your findings in as much detail as possible.

(b) You go on to wonder if you could win (if still letting your guest go first) if some of the stones from the pouch "accidentally go missing." Present your findings in as much detail as possible.

6. You come across a group of finance professors arguing over the recent economic turmoil. The professors are discussing ways to model the various company buy-outs and mergers. One professor, Dr. Rose, has an idea to model the situation with trees, where parent nodes represent companies that have bought out their children nodes. Dr. Rose remarks that she has a beautiful model that combines trees and recursion, two of her favorite topics while taking DM as a student years back. Dr. Rose explains to the other professors, "Suppose you are given a rooted tree T_1 defined as the trivial rooted tree with one vertex, and a rooted tree T_2 also defined as the trivial rooted tree with one vertex. Now, define T_3 to be the rooted tree that has a new root v that is connected to T_1 as its 'left subtree' and that is connected to T_2 as its 'right subtree.' This means that T_3 is a rooted tree with a total of three vertices including one root, a leaf to the left, and a leaf to the right. Continue to define trees T_n in this way, so that in general, tree T_n is constructed using a new root v connected to T_{n-2} as its left subtree and connected to T_{n-1} as its right subtree." She then begins to explain how this can model the business take-overs, but she starts to get confused when she is trying to count the number of vertices in T_n. This is an important point, as the vertices represent the businesses! You swoop in just in time and declare that you will find her a recurrence relation and appropriate initial conditions for the number of vertices V_n in T_n, for all $n \geq 1$. You assure her that you will carefully explain your reasoning, including initial conditions, to prove that you know what you are talking about.